Necessary and sufficient optimality conditions for relaxed and strict control problems of backward systems

Seid BAHLALI
Laboratory of Applied Mathematics, University Med Khider,
Po. Box 145, Biskra 07000, Algeria.
sbahlali@yahoo.fr

December 22, 2008

Abstract

We consider a stochastic control problem where the set of strict (classical) controls is not necessarily convex, and the system is governed by a nonlinear backward stochastic differential equation. By introducing a new approach, we establish necessary as well as sufficient conditions of optimality for two models. The first concerns the relaxed controls, who are measure-valued processes. The second is a particular case of the first and relates to strict control problems.

AMS Subject Classification. 93 Exx

Keywords. Backward stochastic differential equations, Stochastic maximum principle, Strict control, Relaxed control, Adjoint equation, Variational inequality.

1 Introduction

We study a stochastic control problem where the system is governed by a nonlinear backward stochastic differential equation (BSDE for short) of the type

$$\begin{cases} dy_t = b\left(t, y_t^v, z_t^v, v_t\right) dt + z_t^v dW_t, \\ y_T = \xi \end{cases}$$

where b is given maps, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion, defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P}\right)$, satisfying the usual conditions.

The control variable $v = (v_t)$, called strict (classical) control, is an $\mathcal{F}_t$ adapted process with values in some set U of $\mathbb{R}^m$. We denote by $\mathcal{U}$ the class of all strict controls.

The criteria to be minimized, over the set $\mathcal{U}$, has the form

$$J(v) = \mathbb{E}\left[g\left(y_0^v\right) + \int_0^T h\left(t, y_t^v, z_t^v, v_t\right) dt\right],$$

where g and h are given functions and $\left(y_t^v, z_t^v\right)$ is the trajectory of the system controlled by v.

A control $u \in \mathcal{U}$ is called optimal if it satisfies

$$J(u) = \inf_{v \in \mathcal{U}} J(v).$$

Stochastic control problems for backward and forward-backward systems have been studied by many authors including Peng [27], Xu [31], El-Karoui et al [12, 13], Wu [30], Dokuchaev and Zhou [9], Peng and Wu [28], Bahlali and Labed [2], Bahlali [5, 6], Shi and Wu [29], Ji and Zhou [19]. The dynamic programming approach was studied by Fuhrman and Tessitore [16].

Since the strict control domain being nonconvex, then if we use the classical method of spike variation on strict controls, the major difficulty in doing this is that the generator b and the running cost coefficient h depend on two variables y_t and z_t. Then, we can't derive directly the variational inequality, because z_t is hard to handle, there is no convenient pointwise (in t) estimation for it, as opposed to the first variable y_t. To overcome this difficulty, we introduce a new approach which consist to use a bigger new class $\mathcal{R}$ of processes by replacing the U-valued process (v_t) by a $\mathbb{P}(U)$-valued process (q_t), where $\mathbb{P}(U)$ is the space of probability measures on U equipped with the topology of stable convergence. This new class of processes is called relaxed controls and have a richer structure of compacity and convexity. This property of convexity of relaxed controls, enables us to treat the problem with the way of convex perturbation on relaxed controls.

In the relaxed model, the system is governed by the BSDE

$$\begin{cases} dy_t^q = \int_U b\left(t, y_t^q, z_t^q, a\right) q_t(da)\, dt + z_t^q dW_t, \\ y_T^q = \xi. \end{cases}$$

The functional cost to be minimized, over the class $\mathcal{R}$ of relaxed controls, is defined by

$$J(q) = \mathbb{E}\left[g\left(y_0^q\right) + \int_0^T \int_U h\left(t, y_t^q, z_t^q, a\right) q_t(da)\, dt\right].$$

A relaxed control μ is called optimal if it solves

$$J(\mu) = \inf_{q \in \mathcal{R}} J(q).$$

The relaxed control problem is a generalization of the problem of strict controls. Indeed, if $q_t(da) = \delta_{v_t}(da)$ is a Dirac measure concentrated at a

single point $v_t \in U$, then we get a strict control problem as a particular case of the relaxed one.

Our aim in this paper, is to establish necessary as well as sufficient conditions of optimality in the form of global stochastic maximum principle, for both relaxed and strict controls. To achieve this goal, we derive these results as follows.

Firstly, we give the optimality conditions for relaxed controls. The idea is to use the fact that the set of relaxed controls is convex. Then, we establish necessary optimality conditions by using the classical way of the convex perturbation method. More precisely, if we denote by μ an optimal relaxed control and q is an arbitrary element of $\mathcal{R}$, then with a sufficiently small $\theta > 0$ and for each $t \in [0, T]$, we can define a perturbed control as follows

$$\mu_t^\theta = \mu_t + \theta \left(q_t - \mu_t \right).$$

We derive the variational equation from the state equation, and the variational inequality from the inequality

$$0 \leq J\left(\mu^\theta\right) - J\left(\mu\right).$$

By using the fact that the coefficients b and h are linear with respect to the relaxed control variable, necessary optimality conditions are obtained directly in the global form.

To achieve this part of the paper, we prove under minimal additional hypothesis, that these necessary optimality conditions for relaxed controls are also sufficient.

The second main result in the paper characterizes the optimality for strict control processes. It is directly derived from the above result by restricting from relaxed to strict controls. The idea is to replace the relaxed controls by a Dirac measures charging a strict controls. Thus, we reduce the set $\mathcal{R}$ of relaxed controls and we minimize the cost J over the subset $\delta\left(\mathcal{U}\right) = \left\{q \in \mathcal{R} \ / \ q = \delta_v \ ; \ v \in \mathcal{U}\right\}$. Necessary optimality conditions for strict controls are then obtained directly from those of relaxed one. Finally, we prove that these necessary conditions becomes sufficient, without imposing neither the convexity of U nor that of the Hamiltonian H in v.

The paper is organized as follows. In Section 2, we formulate the strict and relaxed control problems and give the various assumptions used throughout the paper. Section 3 is devoted to study the relaxed control problems and we establish necessary as well as sufficient conditions of optimality for relaxed controls. In the last section, we derive directly from the results of Section 3, the optimality conditions for strict controls.

Along this paper, we denote by C some positive constant and we need the following matrix notations. We denote by $\mathcal{M}_{n\times d}\left(\mathbb{R}\right)$ the space of $n \times d$ real matrices and by $\mathcal{M}_{n\times n}^d\left(\mathbb{R}\right)$ the linear space of vectors $M = (M_1, ..., M_d)$ where $M_i \in \mathcal{M}_{n\times n}\left(\mathbb{R}\right)$.

For any $M, N \in \mathcal{M}^d_{n\times n}(\mathbb{R}),\ L, S \in \mathcal{M}_{n\times d}(\mathbb{R}),\ Q \in \mathcal{M}_{n\times n}(\mathbb{R}),\ \alpha, \beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^d$, we use the following notations

$\alpha\beta = \sum_{i=1}^{n} \alpha_i \beta_i \in \mathbb{R}$ is the product scalar in $\mathbb{R}^n$;

$LS = \sum_{i=1}^{d} L_i S_i \in \mathbb{R}$, where L_i and S_i are the i^{th} columns of L and S;

$ML = \sum_{i=1}^{d} M_i L_i \in \mathbb{R}^n$;

$M\alpha\gamma = \sum_{i=1}^{d} (M_i\alpha)\gamma_i \in \mathbb{R}^n$;

$MN = \sum_{i=1}^{d} M_i N_i \in \mathcal{M}_{n\times n}(\mathbb{R})$;

$MQN = \sum_{i=1}^{d} M_i Q N_i \in \mathcal{M}_{n\times n}(\mathbb{R})$;

$MQ\gamma = \sum_{i=1}^{d} M_i Q \gamma_i \in \mathcal{M}_{n\times n}(\mathbb{R})$.

We denote by L^* the transpose of the matrix L and $M^* = (M_1^*, ..., M_d^*)$.

2 Formulation of the problem

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P}\right)$ be a filtered probability space satisfying the usual conditions, on which a d-dimensional Brownian motion $W = (W_t)_{t\geq 0}$ is defined. We assume that $(\mathcal{F}_t)$ is the $\mathcal{P}$- augmentation of the natural filtration of W.

Let T be a strictly positive real number and U a subset of $\mathbb{R}^m$.

2.1 The strict control problem

Definition 1 *An admissible strict control is an $\mathcal{F}_t-$ adapted process $v = (v_t)$ with values in U such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |v_t|^2\right] < \infty.$$

We denote by $\mathcal{U}$ the set of all admissible strict controls.

For any $v \in \mathcal{U}$, we consider the following controlled BSDE

$$\begin{cases} dy_t^v = b\left(t, y_t^v, z_t^v, v_t\right)dt + z_t^v dW_t, \\ y_T^v = \xi, \end{cases} \tag{1}$$

where $b : [0,T] \times \mathbb{R}^n \times \mathcal{M}_{n\times d}(\mathbb{R}) \times U \longrightarrow \mathbb{R}^n$ and ξ is an n-dimensional $\mathcal{F}_T$-measurable random variable such that

$$\mathbb{E}|\xi|^2 < \infty.$$

The criteria to be minimized is defined from $\mathcal{U}$ into $\mathbb{R}$ by

$$J(v) = \mathbb{E}\left[g\left(y_0^v\right) + \int_0^T h\left(t, y_t^v, z_t^v, v_t\right) dt\right], \tag{2}$$

where,

$$\begin{aligned} &g : \mathbb{R}^n \longrightarrow \mathbb{R}, \\ &h : [0,T] \times \mathbb{R}^n \times \mathcal{M}_{n\times d}(\mathbb{R}) \times U \longrightarrow \mathbb{R}. \end{aligned}$$

A strict control u is called optimal if it satisfies

$$J(u) = \inf_{v\in\mathcal{U}} J(v). \tag{3}$$

We assume that

$$b, g \text{ and } h \text{ are continuously differentiable with respect to } (y,z). \tag{4}$$

They and all their derivatives with respect to (y,z) are continuous in (y,z,v). They are bounded by $C\ (1 + |y| + |z| + |v|)$ and their derivatives with respect to (y,z) are continuous and uniformly bounded.

Under the above hypothesis, for every $v \in U$, equation (1) has a unique strong solution and the functional cost J is well defined from $\mathcal{U}$ into $\mathbb{R}$.

2.2 The relaxed model

The idea for relaxed the strict control problem defined above is to embed the set U of strict controls into a wider class which gives a more suitable topological structure. In the relaxed model, the U-valued process v is replaced by a $\mathbb{P}(U)$-valued process q, where $\mathbb{P}(U)$ denotes the space of probability measure on U equipped with the topology of stable convergence.

Definition 2 *A relaxed control $(q_t)_t$ is a $\mathbb{P}(U)$-valued process, progressively measurable with respect to $(\mathcal{F}_t)_t$ and such that for each t, $1_{]0,t]}.q$ is $\mathcal{F}_t$-measurable.*

We denote by $\mathcal{R}$ the set of all relaxed controls.

Remark 3 *The set of strict controls is embedded into the set of relaxed controls by the mapping*

$$f : v \longmapsto f_v(dt, da) = dt\delta_{v_t}(da),$$

where δ_v is the atomic measure concentrated at a single point v.

For more details on relaxed controls, see [1], [3], [4], [5], [10], [14], [21], [23], [24].

For any $q \in \mathcal{R}$, we consider the following relaxed BSDE

$$\begin{cases} dy_t^q = \int_U b\left(t, y_t^q, z_t^q, a\right) q_t(da)\, dt + z_t^q dW_t, \\ y_T^q = \xi. \end{cases} \tag{5}$$

The expected cost to be minimized, in the relaxed model, is defined from $\mathcal{R}$ into $\mathbb{R}$ by

$$J(q) = \mathbb{E}\left[g\left(y_0^q\right) + \int_0^T \int_U h\left(t, y_t^q, z_t^q, a\right) q_t(da)\, dt \right]. \tag{6}$$

A relaxed control μ is called optimal if it solves

$$J(\mu) = \inf_{q \in \mathcal{R}} J(q). \tag{7}$$

Remark 4 *If we put*

$$\overline{b}\left(t, y_t^q, z_t^q, q_t\right) = \int_U b\left(t, y_t^q, z_t^q, a\right) q_t(da),$$
$$\overline{h}\left(t, y_t^q, z_t^q, q_t\right) = \int_U h\left(t, y_t^q, z_t^q, a\right) q_t(da).$$

Then, equation (5) *becomes*

$$\begin{cases} dy_t^q = \overline{b}\left(t, y_t^q, z_t^q, q_t\right) dt + z_t^q dW_t, \\ y_T^q = \xi. \end{cases} \tag{5'}$$

With a functional cost given by

$$J(q) = \mathbb{E}\left[g\left(y_0^q\right) + \int_0^T \overline{h}\left(t, y_t^q, z_t^q, q_t\right) dt \right].$$

Hence, by introducing relaxed controls, we have replaced U *by a larger space* $\mathbb{P}(U)$*. We have gained the advantage that* $\mathbb{P}(U)$ *is both compact and convex. Furthermore, the new coefficients of equation* $(5')$ *and the running cost are linear with respect to the relaxed control variable.*

Remark 5 *The coefficient* $\overline{b}$ *(defined in the above remark) check respectively the same assumptions as* b*. Then, under assumptions* (4)*,* $\overline{b}$ *is uniformly Lipschitz and with linear growth. Then by classical results on BSDEs, for every* $q \in \mathcal{R}$ *equation* $(5')$ *admits a unique strong solution. Consequently, for every* $q \in \mathcal{R}$ *equation* (5) *has a unique strong solution.*

On the other hand, It is easy to see that $\overline{h}$ *checks the same assumptions as* h*. Then, the functional cost* J *is well defined from* $\mathcal{R}$ *into* $\mathbb{R}$*.*

Remark 6 *If* $q_t = \delta_{v_t}$ *is an atomic measure concentrated at a single point* $v_t \in U$*, then for each* $t \in [0, T]$ *we have*

$$\int_U b\left(t, y_t^q, z_t^q, a\right) q_t(da) = \int_U b\left(t, y_t^q, z_t^q, a\right) \delta_{v_t}(da) = b\left(t, y_t^q, z_t^q, v_t\right),$$
$$\int_U h\left(t, y_t^q, z_t^q, a\right) q_t(da) = \int_U h\left(t, y_t^q, z_t^q, a\right) \delta_{v_t}(da) = h\left(t, y_t^q, z_t^q, v_t\right).$$

In this case $\left(y^q, z^q\right) = \left(y^v, z^v\right)$*,* $J(q) = J(v)$ *and we get a strict control problem. So the problem of strict controls* $\{(1), (2), (3)\}$ *is a particular case of relaxed control problem* $\{(5), (6), (7)\}$*.*

3 Necessary and sufficient optimality conditions for relaxed controls

In this section, we study the problem $\{(5),(6),(7)\}$ and we establish necessary as well as sufficient conditions of optimality for relaxed controls.

3.1 Preliminary results

Since the set $\mathcal{R}$ is convex, then the classical way to derive necessary optimality conditions for relaxed controls is to use the convex perturbation method. More precisely, let μ be an optimal relaxed control and (y_t^μ, z_t^μ) the solution of (5) controlled by μ. Then, we can define a perturbed relaxed control as follows

$$\mu_t^\theta = \mu_t + \theta\left(q_t - \mu_t\right), \tag{8}$$

where, $\theta > 0$ is sufficiently small and q is an arbitrary element of $\mathcal{R}$.

Denote by $\left(y_t^\theta, z_t^\theta\right)$ the solution of (5) associated with μ^θ.

From optimality of μ, the variational inequality will be derived from the fact that

$$0 \leq J\left(\mu^\theta\right) - J\left(\mu\right). \tag{9}$$

For this end, we need the following classical lemmas.

Lemma 7 *Under assumptions* (4)*, we have*

$$\lim_{\theta\to 0}\left[\sup_{0\leq t\leq T}\mathbb{E}\left|y_t^\theta - y_t^\mu\right|^2\right] = 0, \tag{10}$$

$$\lim_{\theta\to 0}\mathbb{E}\int_0^T\left|z_t^\theta - z_t^\mu\right|^2 dt = 0. \tag{11}$$

Proof. Applying Itô's formula to $\left(y_t^\theta - y_t^\mu\right)^2$, we have

$$\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t^\mu\right|^2 + \mathbb{E}\int_t^T\left|z_s^\theta - z_s^\mu\right|^2 ds \\
&= 2\mathbb{E}\int_t^T\left|\left(y_s^\theta - y_s^\mu\right)\left[\int_U b\left(s, y_s^\theta, z_s^\theta, a\right)\mu_s^\theta(da) - \int_U b\left(s, y_s^\mu, z_s^\mu, a\right)\mu_s(da)\right]\right| ds.
\end{aligned}$$

From the Young formula, for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t^\mu\right|^2 + \mathbb{E}\int_t^T\left|z_s^\theta - z_s^\mu\right|^2 ds \\
&\leq \frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|y_s^\theta - y_s^\mu\right|^2 ds \\
&+ \varepsilon\mathbb{E}\int_t^T\left|\int_U b\left(s, y_s^\theta, z_s^\theta, a\right)\mu_s^\theta(da) - \int_U b\left(s, y_s^\mu, z_s^\mu, a\right)\mu_s(da)\right|^2 ds.
\end{aligned}$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t^\mu\right|^2 + \mathbb{E}\int_t^T \left|z_s^\theta - z_s^\mu\right|^2 ds \\
&\leq \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|y_s^\theta - y_s^\mu\right|^2 ds \\
&+ C\varepsilon\mathbb{E}\int_t^T \left|\int_U b\left(s, y_s^\theta, z_s^\theta, a\right)\mu_s^\theta(da) - \int_U b\left(s, y_s^\theta, z_s^\theta, a\right)\mu_s(da)\right|^2 ds \\
&+ C\varepsilon\mathbb{E}\int_t^T \left|\int_U b\left(s, y_s^\theta, z_s^\theta, a\right)\mu_s(da) - \int_U b\left(s, y_s^\mu, z_s^\theta, a\right)\mu_s(da)\right|^2 ds \\
&+ C\varepsilon\mathbb{E}\int_t^T \left|\int_U b\left(s, y_s^\mu, z_s^\theta, a\right)\mu_s(da) - \int_U b\left(s, y_s^\mu, z_s^\mu, a\right)\mu_s(da)\right| ds.
\end{aligned}
$$

By the definition of μ_t^θ, we have

$$
\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t^\mu\right|^2 + \mathbb{E}\int_t^T \left|z_s^\theta - z_s^\mu\right|^2 ds \\
&\leq \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|y_s^\theta - y_s^\mu\right|^2 ds \\
&+ C\varepsilon\theta^2\mathbb{E}\int_t^T \left|\int_U b\left(s, y_s^\theta, z_s^\theta, a\right)q_s(da) - \int_U b\left(s, y_s^\theta, z_s^\theta, a\right)\mu_s(da)\right|^2 ds \\
&+ C\varepsilon\mathbb{E}\int_t^T \left|\int_U b\left(s, y_s^\theta, z_s^\theta, a\right)\mu_s(da) - \int_U b\left(s, y_s^\mu, z_s^\theta, a\right)\mu_s(da)\right|^2 ds \\
&+ C\varepsilon\mathbb{E}\int_t^T \left|\int_U b\left(s, y_s^\mu, z_s^\theta, a\right)\mu_s(da) - \int_U b\left(s, y_s^\mu, z_s^\mu, a\right)\mu_s(da)\right|^2 ds.
\end{aligned}
$$

Since b is uniformly Lipschitz with respect to y, z, then

$$
\begin{aligned}
\mathbb{E}\left|y_t^\theta - y_t^\mu\right|^2 + \mathbb{E}\int_t^T \left|z_s^\theta - z_s^\mu\right|^2 ds \leq \left(\frac{1}{\varepsilon} + C\,\varepsilon\right)\mathbb{E}\int_t^T \left|y_s^\theta - y_s^\mu\right|^2 ds \\
+ C\varepsilon\mathbb{E}\int_t^T \left|z_s^\theta - z_s^\mu\right|^2 ds + C\varepsilon\theta^2.
\end{aligned}
$$

Choose $\varepsilon = \dfrac{1}{2C}$, then we have

$$
\mathbb{E}\left|y_t^\theta - y_t^\mu\right|^2 + \frac{1}{2}\mathbb{E}\int_t^T \left|z_s^\theta - z_s^\mu\right|^2 ds \leq \left(2C + \frac{1}{2}\right)\mathbb{E}\int_t^T \left|y_s^\theta - y_s^\mu\right|^2 ds + C\varepsilon\theta^2.
$$

From the above inequality, we derive two inequalities

$$
\mathbb{E}\left|y_t^\theta - y_t^\mu\right|^2 \leq \left(2C + \frac{1}{2}\right)\mathbb{E}\int_t^T \left|y_s^\theta - y_s^\mu\right|^2 ds + C\varepsilon\theta^2, \tag{12}
$$

$$\mathbb{E}\int_t^T \left|z_s^\theta - z_s^\mu\right|^2 ds \le (4C+1)\,\mathbb{E}\int_t^T \left|y_s^\theta - y_s^\mu\right|^2 ds + 2C\varepsilon\theta^2. \tag{13}$$

By using (12), Gronwall's lemma and Bukholder-Davis-Gundy inequality, we obtain (10). Finally, (11) is derived from (10) and (13). ■

Lemma 8 *Let $\widetilde{y}_t$ be the solution of the following linear equation (called variational equation)*

$$\left\{\begin{array}{ll} d\widetilde{y}_t = & \int_U \left[b_y\left(t, y_t^\mu, z_t^\mu, a\right)\widetilde{y}_t + b_z\left(t, y_t^\mu, z_t^\mu, a\right)\widetilde{z}_t\right]\mu_t(da)\,dt \\ & +\left[\int_U b\left(t, y_t^\mu, z_t^\mu, a\right) q_t(da) - \int_U b\left(t, y_t^\mu, z_t^\mu, a\right)\mu_t(da)\right] dt + \widetilde{z}_t dW_t, \\ \widetilde{y}_T = & 0. \end{array}\right. \tag{14}$$

Then, the following estimations hold

$$\lim_{\theta\to 0}\mathbb{E}\left|\frac{y_t^\theta - y_t^\mu}{\theta} - \widetilde{y}_t\right|^2 = 0, \tag{15}$$

$$\lim_{\theta\to 0}\mathbb{E}\int_0^T \left|\frac{z_t^\theta - z_t^\mu}{\theta} - \widetilde{z}_t\right|^2 dt = 0. \tag{16}$$

Proof. For simplicity, we put

$$Y_t^\theta = \frac{y_t^\theta - y_t^\mu}{\theta} - \widetilde{y}_t, \tag{17}$$

$$Z_t^\theta = \frac{z_t^\theta - z_t^\mu}{\theta} - \widetilde{z}_t. \tag{18}$$

$$\Lambda_t^\theta(a) = \left(t, y_t^\mu + \lambda\theta\left(Y_t^\theta + \widetilde{y}_t\right), z_t^\mu + \lambda\theta\left(Z_t^\theta + \widetilde{z}_t\right), a\right). \tag{19}$$

By (17) and (18) we have the following BSDE

$$\left\{\begin{array}{l} dY_t^\theta = \left(F_t^y Y_t^\theta dt + F_t^y Z_t^\theta - \gamma_t^\theta\right) dt + Z_t^\theta dW_t, \\ Y_T^\theta = 0, \end{array}\right.$$

where,

$$F_t^y = -\int_0^1 \int_U b_y\left(\Lambda_t^\theta(a)\right)\mu_t(da)\,d\lambda,$$
$$F_t^z = -\int_0^1 \int_U b_z\left(\Lambda_t^\theta(a)\right)\mu_t(da)\,d\lambda,$$

and γ_t^θ is given by

$$\gamma_t^\theta = \int_t^T \int_U \left[b_y\left(\Lambda_s^\theta(a)\right)\left(y_s^\theta - y_s^\mu\right) + b_z\left(\Lambda_s^\theta(a)\right)\left(z_s^\theta - z_s^\mu\right)\right] q_s(da)\,ds$$
$$-\int_t^T \int_U \left[b_y\left(\Lambda_s^\theta(a)\right)\left(y_s^\theta - y_s^\mu\right) + b_z\left(\Lambda_s^\theta(a)\right)\left(z_s^\theta - z_s^\mu\right)\right] \mu_s(da)\,ds.$$

Since b_y and b_z are bounded, then

$$\mathbb{E}\left|\gamma_t^\theta\right|^2 \leq C\mathbb{E}\int_t^T \left|y_s^\theta - y_s^\mu\right|^2 ds + C\mathbb{E}\int_t^T \left|z_s^\theta - z_s^\mu\right|^2 ds$$

By (10) and (11), we get

$$\lim_{\theta\to 0}\mathbb{E}\left|\gamma_t^\theta\right|^2 = 0. \tag{20}$$

Applying Itô's formula to $\left(Y_t^\theta\right)^2$, we get

$$\mathbb{E}\left|Y_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|Z_s^\theta\right|^2 ds = \mathbb{E}\left|Y_T^\theta\right|^2 + 2\mathbb{E}\int_t^T \left|Y_s^\theta\left(F_s^y Y_s^\theta + F_s^z Z_s^\theta - \gamma_s^\theta\right)\right| ds.$$

By using the Young formula, for every $\varepsilon > 0$, we have

$$\begin{aligned}
\mathbb{E}\left|Y_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|Z_s^\theta\right|^2 ds &\leq \mathbb{E}\left|Y_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|Y_s^\theta\right|^2 ds + \varepsilon\mathbb{E}\int_t^T \left|\left(F_s^y Y_s^\theta + F_s^z Z_s^\theta - \gamma_s^\theta\right)\right|^2 ds \\
&\leq \mathbb{E}\left|Y_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|Y_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|F_s^y Y_s^\theta\right|^2 ds \\
&\quad + C\varepsilon\mathbb{E}\int_t^T \left|F_s^z Z_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|\gamma_s^\theta\right|^2 ds.
\end{aligned}$$

Since F_t^y and F_t^z are bounded, then

$$\mathbb{E}\left|Y_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|Z_s^\theta\right|^2 ds \leq \left(\frac{1}{\varepsilon} + C\ \varepsilon\right)\mathbb{E}\int_t^T \left|Y_s^\theta\right|^2 ds + C\ \varepsilon\mathbb{E}\int_t^T \left|Z_s^\theta\right|^2 ds + C\ \varepsilon\mathbb{E}\int_t^T \left|\gamma_s^\theta\right|^2 ds,$$

Choose $\varepsilon = \dfrac{1}{2C}$, then we have

$$\mathbb{E}\left|Y_t^\theta\right|^2 + \frac{1}{2}\mathbb{E}\int_t^T \left|Z_s^\theta\right|^2 ds \leq \left(2C + \frac{1}{2}\right)\mathbb{E}\int_t^T \left|Y_s^\theta\right|^2 ds + C\ \varepsilon\mathbb{E}\int_t^T \left|\gamma_s^\theta\right|^2 ds.$$

From the above inequality, we deduce two inequalities

$$\mathbb{E}\left|Y_t^\theta\right|^2 \leq \left(2C + \frac{1}{2}\right)\mathbb{E}\int_t^T \left|Y_s^\theta\right|^2 ds + C\ \varepsilon\mathbb{E}\int_t^T \left|\gamma_s^\theta\right|^2 ds, \tag{21}$$

$$\mathbb{E}\int_t^T \left|Z_s^\theta\right|^2 ds \leq (4C+1)\,\mathbb{E}\int_t^T \left|Y_s^\theta\right|^2 ds + 2C\ \varepsilon\mathbb{E}\int_t^T \left|\gamma_s^\theta\right|^2 ds. \tag{22}$$

By using (20), (21) and Gronwall's lemma, we obtain (15). Finally (16) is derived from (15), (20) and (21). ■

Lemma 9 *Let μ be an optimal control minimizing the functional J over $\mathcal{R}$ and (y_t^μ, z_t^μ) the solution of (5) associated with μ. Then for any $q \in \mathcal{R}$, we have*

$$\begin{aligned} 0 \leq\ & \mathbb{E}\left[g_y\left(y_0^\mu\right)\widetilde{y}_0\right] \qquad (23)\\ & + \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^\mu, z_t^\mu, a\right) q_t\left(da\right) - \int_U h\left(t, y_t^\mu, z_t^\mu, a\right)\mu_t\left(da\right)\right] dt \\ & + \mathbb{E}\int_0^T \int_U h_y\left(t, y_t^\mu, z_t^\mu, a\right)\mu_t\left(da\right)\widetilde{y}_t dt + \mathbb{E}\int_0^T \int_U h_z\left(t, y_t^\mu, z_t^\mu, a\right)\mu_t\left(da\right)\widetilde{z}_t dt. \end{aligned}$$

Proof. Let μ be an optimal relaxed control minimizing the cost J over $\mathcal{R}$, then from (9) we have

$$\begin{aligned} 0 & \leq \mathbb{E}\left[g\left(y_0^\theta\right) - g\left(y_0^\mu\right)\right] \\ & \quad + \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^\theta, z_t^\theta, a\right)\mu_t^\theta\left(da\right) - \int_U h\left(t, y_t^\mu, z_t^\mu, a\right)\mu_t\left(da\right)\right] dt \\ & = \mathbb{E}\left[g\left(y_0^\theta\right) - g\left(y_0^\mu\right)\right] \\ & \quad + \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^\theta, z_t^\theta, a\right)\mu_t^\theta\left(da\right) - \int_U h\left(t, y_t^\theta, z_t^\theta, a\right)\mu_t\left(da\right)\right] dt \\ & \quad + \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^\theta, z_t^\theta, a\right)\mu_t\left(da\right) - \int_U h\left(t, y_t^\mu, z_t^\mu, a\right)\mu_t\left(da\right)\right] dt. \end{aligned}$$

From the definition of μ^θ, we get

$$\begin{aligned} 0 & \leq \mathbb{E}\left[g\left(y_0^\theta\right) - g\left(y_0^\mu\right)\right] \\ & \quad + \theta\mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^\theta, z_t^\theta, a\right) q_t\left(da\right) - \int_U h\left(t, y_t^\theta, z_t^\theta, a\right)\mu_t\left(da\right)\right] dt \\ & \quad + \mathbb{E}\int_0^T \int_U \left[h\left(t, y_t^\theta, z_t^\theta, a\right) - h\left(t, y_t^\mu, z_t^\mu, a\right)\right]\mu_t\left(da\right) dt. \end{aligned}$$

Then,

$$\begin{aligned} 0 \leq\ & \mathbb{E}\int_0^1 \left[g_y\left(y_0^\mu + \lambda\theta\left(\widetilde{y}_0 + Y_0^\theta\right)\right)\widetilde{y}_0\right] d\lambda \qquad (24)\\ & + \mathbb{E}\int_0^T \int_0^1 \int_U \left[h_y\left(\Lambda_t^\theta\left(a\right)\right)\widetilde{y}_t + h_z\left(\Lambda_t^\theta\left(a\right)\right)\widetilde{z}_t\right]\mu_t\left(da\right) d\lambda dt \\ & + \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^\mu, z_t^\mu, a\right) q_t\left(da\right) - \int_U h\left(t, y_t^\mu, z_t^\mu, a\right)\mu_t\left(da\right)\right] dt + \rho_t^\theta, \end{aligned}$$

where ρ_t^θ is given by

$$\rho_t^\theta = \mathbb{E}\int_0^1 \left[g_y\left(y_0^\mu + \lambda\theta\left(\widetilde{y}_0 + Y_0^\theta\right)\right) Y_0^\theta\right] d\lambda$$
$$+\mathbb{E}\int_0^T\int_0^1\int_U \left[h_y\left(\Lambda_t^\theta(a)\right)\left(y_t^\theta - y_t^\mu\right) + h_z\left(\Lambda_t^\theta(a)\right)\left(z_t^\theta - z_t^\mu\right)\right]\mu_t(da)\, d\lambda dt$$
$$+\mathbb{E}\int_0^T\int_0^1\int_U \left[h_y\left(\Lambda_t^\theta(a)\right) Y_t^\theta + h_z\left(\Lambda_t^\theta(a)\right) Z_t^\theta\right]\mu_t(da)\, d\lambda dt.$$

Since the derivatives g_y, h_y, h_z are bounded, then by using the Cauchy-Schwartz inequality, we have

$$\rho_t^\theta = \left(\mathbb{E}\left|Y_0^\theta\right|^2\right)^{1/2} + \left(\mathbb{E}\int_0^T \left|y_t^\theta - y_t^\mu\right|^2 dt\right)^{1/2} + \left(\mathbb{E}\int_0^T \left|z_t^\theta - z_t^\mu\right|^2 dt\right)^{1/2}$$
$$+\left(\mathbb{E}\int_0^T \left|Y_t^\theta\right|^2 dt\right)^{1/2} + \left(\mathbb{E}\int_0^T \left|Z_t^\theta\right|^2 dt\right)^{1/2}.$$

By (10), (11), (15) and (16), we get

$$\lim_{\theta\to 0}\rho_t^\theta = 0.$$

Finally, by letting θ go to 0 in (24), the proof is completed. ■

3.2 Necessary optimality conditions for relaxed controls

Starting from the variational inequality (23), we can now state necessary optimality conditions for the relaxed control problem $\{(5), (6), (7)\}$ in the global form.

Theorem 10 *(Necessary optimality conditions for relaxed controls) Let μ be an optimal relaxed control minimizing the functional J over $\mathcal{R}$ and (y_t^μ, z_t^μ) the solution of (5) associated with μ. Then, there an unique adapted process p^μ, which is the solution of the forward stochastic equation (called adjoint equation),*

$$\begin{cases} -dp_t^\mu = \mathcal{H}_y\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) dt + \mathcal{H}_z\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) dW_t, \\ p_0^\mu = g_y(y_0^\mu), \end{cases} \tag{25}$$

such that

$$\mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) \geq \mathcal{H}\left(t, y_t^\mu, z_t^\mu, q_t, p_t^\mu\right),\ \forall q_t \in \mathbb{P}(U),\ ae\ , as. \tag{26}$$

where the Hamiltonian $\mathcal{H}$ is defined from $[0,T]\times\mathbb{R}^n\times\mathcal{M}_{n\times d}(\mathbb{R})\times\mathbb{R}^n\times\mathbb{P}(U)$ into $\mathbb{R}$ by

$$\mathcal{H}(t,y,z,p,q) = p\int_U b(t,y,z,a)\, q_t(da) - \int_U h(t,y,z,a)\, q_t(da).$$

Proof. Since $p_0^\mu = g_y(y_0^\mu)$, then (23) becomes

$$0 \leq \mathbb{E}\left[p_0^\mu \widetilde{y}_0\right] + \mathbb{E}\int_0^T \int_U \left[h_y\left(t, y_t^\mu, z_t^\mu, a\right) \widetilde{y}_t + h_z\left(t, y_t^\mu, z_t^\mu, a\right) \widetilde{z}_t\right] \mu_t\left(da\right) dt \qquad (27)$$
$$+ \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^\mu, z_t^\mu, a\right) q_t\left(da\right) - \int_U h\left(t, y_t^\mu, z_t^\mu, a\right) \mu_t\left(da\right)\right] dt.$$

By applying Itô's formula to $(p_t^\mu \widetilde{y}_t)$, we have

$$\mathbb{E}\left[p_0^\mu \widetilde{y}_0\right] = -\mathbb{E}\left[\int_0^T \int_U h_y\left(t, y_t^\mu, z_t^\mu, a\right) \mu_t\left(da\right) \widetilde{y}_t + \int_U h_z\left(t, y_t^\mu, z_t^\mu, a\right) \mu_t\left(da\right) \widetilde{z}_t\right] dt$$
$$+ \mathbb{E}\int_0^T p_t^\mu \left[\int_U b\left(t, y_t^\mu, z_t^\mu, a\right) \mu_t\left(da\right) - \int_U b\left(t, y_t^\mu, z_t^\mu, a\right) q_t\left(da\right)\right] dt$$

Then for every $q \in \mathcal{R}$, (27) becomes

$$0 \leq \mathbb{E}\int_0^T \left[\mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) - \mathcal{H}\left(t, y_t^\mu, z_t^\mu, q_t, p_t^\mu\right)\right] dt. \qquad (28)$$

The theorem is proved. ■

3.3 Sufficient optimality conditions for relaxed controls

In this subsection, we study when necessary optimality conditions (26) becomes sufficient. We recall assumptions (4) and the adjoints equation (25). For any $q \in \mathcal{R}$, we denote by (y^q, z^q) the solution of equation (5) controlled by q.

Theorem 11 *(Sufficient optimality conditions for relaxed controls).Assume that g and the function $(y, z) \longmapsto \mathcal{H}(t, y, z, q, p)$ is concave. Then, μ is an optimal solution of the relaxed control problem $\{(5), (6), (7)\}$, if it satisfies* (26).

Proof. Let μ be an arbitrary element of $\mathcal{R}$ (candidate to be optimal). For any $q \in \mathcal{R}$, we have

$$J\left(q\right) - J\left(\mu\right) = \mathbb{E}\left[g\left(y_0^q\right) - g\left(y_0^\mu\right)\right]$$
$$+ \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^q, z_t^q, a\right) q_t\left(da\right) - \int_U h\left(t, y_t^\mu, z_t^\mu, a\right) \mu_t\left(da\right)\right] dt.$$

Since g is convex, then

$$g\left(y_0^q\right) - g\left(y_0^\mu\right) \geq g_y\left(y_0^\mu\right)\left(y_0^q - y_0^\mu\right).$$

Thus,

$$J\left(q\right) - J\left(\mu\right) \geq \mathbb{E}\left[g_y\left(y_0^\mu\right)\left(y_0^q - y_0^\mu\right)\right]$$
$$+ \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^q, z_t^q, a\right) q_t\left(da\right) - \int_U h\left(t, y_t^\mu, z_t^\mu, a\right) \mu_t\left(da\right)\right] dt.$$

We remark from (25) that

$$p_0^\mu = g_y\left(y_0^\mu\right).$$

Then, we have

$$J\left(q\right) - J\left(\mu\right) \geq \mathbb{E}\left[p_0^\mu\left(y_0^q - y_0^\mu\right)\right] + \mathbb{E}\int_0^T \left[\int_U h\left(t, y_t^q, z_t^q, a\right) q_t\left(da\right) - \int_U h\left(t, y_t^\mu, z_t^\mu, a\right) \mu_t\left(da\right)\right] dt.$$

By applying Itô's formula to $p_t^\mu\left(y_t^q - y_t^\mu\right)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[p_0^\mu\left(y_0^q - y_0^\mu\right)\right] &= \mathbb{E}\int_0^T \left[\mathcal{H}_y\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right)\left(y_t^q - y_t^\mu\right) + \mathcal{H}_z\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right)\left(z_t^q - z_t^\mu\right)\right] dt \\
&\quad + \mathbb{E}\int_0^T p_t^\mu\left[\int_U b\left(t, y_t^\mu, z_t^\mu, a\right)\mu_t\left(da\right) - \int_U b\left(t, y_t^q, z_t^q, a\right) q_t\left(da\right)\right] dt
\end{aligned}$$

Then,

$$\begin{aligned}
&J\left(q\right) - J\left(\mu\right) \\
&\geq \mathbb{E}\int_0^T \left[\mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) - \mathcal{H}\left(t, y_t^q, z_t^q, q_t, p_t^\mu\right)\right] dt \\
&+ \mathbb{E}\int_0^T \mathcal{H}_y\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right)\left(y_t^q - y_t^\mu\right) dt \\
&+ \mathbb{E}\int_0^T \mathcal{H}_z\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right)\left(z_t^q - z_t^\mu\right) dt.
\end{aligned} \tag{29}$$

Since $\mathcal{H}$ is concave in (y, z) and linear in μ, then by using the Clarke generalized gradient of $\mathcal{H}$ evaluated at (y_t, z_t, μ_t) and the necessary optimality conditions (26), it follows by [33, Lemmas 2.2 and 2.3] that

$$\begin{aligned}
\mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) - \mathcal{H}\left(t, y_t^q, z_t^q, q_t, p_t^\mu\right) &\geq -\mathcal{H}_y\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right)\left(y_t^q - y_t^\mu\right) \\
&\quad - \mathcal{H}_z\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right)\left(z_t^q - z_t^\mu\right)
\end{aligned}$$

Or equivalently,

$$\begin{aligned}
0 &\leq \mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) - \mathcal{H}\left(t, y_t^q, z_t^q, q_t, p_t^\mu\right) + \mathcal{H}_y\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right)\left(y_t^q - y_t^\mu\right) \\
&+ \mathcal{H}_z\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right)\left(z_t^q - z_t^\mu\right)
\end{aligned}$$

Then from (29), we get

$$J\left(q\right) - J\left(\mu\right) \geq 0.$$

The theorem is proved. ■

4 Necessary and sufficient optimality conditions for strict controls

In this section, we study the strict control problem $\{(1),(2),(3)\}$ and from the results of section 3, we derive the optimality conditions for strict controls.

Throughout this section we suppose moreover that

$$U \text{ is compact.} \tag{30}$$
$$b \text{ and } h \text{ are bounded.}$$

Consider the following subset of $\mathcal{R}$

$$\delta(\mathcal{U}) = \{q \in \mathcal{R} \ / \ q = \delta_v \ ; \ v \in \mathcal{U}\}. \tag{31}$$

The set $\delta(\mathcal{U})$ is the collection of all relaxed controls in the form of Dirac measure charging a strict control.

Denote by $\delta(U)$ the action set of all relaxed controls in $\delta(\mathcal{U})$.

If $q \in \delta(\mathcal{U})$, then $q = \delta_v$ with $v \in \mathcal{U}$. In this case we have for each t, $q_t \in \delta(U)$ and $q_t = \delta_{v_t}$.

We equipped $\mathbb{P}(U)$ with the topology of stable convergence. Since U is compact, then with this topology $\mathbb{P}(U)$ is a compact metrizable space. The stable convergence is required for bounded measurable functions $f(t,a)$ such that for each fixed $t \in [0,T]$, $f(t,.)$ is continuous (Instead of functions bounded and continuous with respect to the pair (t,a) for the weak topology). The space $\mathbb{P}(U)$ is equipped with its Borel σ-field, which is the smallest σ-field such that the mapping $q \longmapsto \displaystyle\int f(s,a)\, q(ds,da)$ are measurable for any bounded measurable function f, continuous with respect to a. For more details, see Jacod and Memin [29] and El Karoui et al [16].

This allows us to summarize some of lemmas that we will be used in the sequel.

Lemma 12 *(Chattering Lemma). Let q be a predictable process with values in the space of probability measures on U. Then there exists a sequence of predictable processes $(u^n)_n$ with values in U such that*

$$dtq_t^n(da) = dt\delta_{u_t^n}(da) \underset{n\longrightarrow\infty}{\longrightarrow} dtq_t(da) \ \text{ stably}, \ \ \mathcal{P}-a.s. \tag{32}$$

where $\delta_{u_t^n}$ is the Dirac measure concentrated at a single point u_t^n of U.

Proof. See El karoui et al [14]. ■

Lemma 13 *Let q be a relaxed control and $(u^n)_n$ be a sequence of strict controls such that () holds. Then for any function $f : [0,T] \times U \to \mathbb{R}$, measurable in t and continuous in a, we have*

$$\int_U f(t,a)\,\delta_{u_t^n}(da) \underset{n\longrightarrow\infty}{\longrightarrow} \int_U f(t,a)\,q_t(da) \ ; \ dt-a.e \tag{33}$$

Proof. By (29) and the definition of the stable convergence (See Jacod-Memin [21]), for any bounded function $f(t,a)$ measurable in t and continuous in a, we have

$$\int_0^T \int_U f(t,a)\, \delta_{u_t^n}(da) \underset{n\longrightarrow\infty}{\longrightarrow} \int_0^T \int_U f(t,a)\, q_t(da).$$

Put

$$g(s,a) = 1_{[0,t]}(s)\, f(s,a).$$

It's clear that

$$\int_0^T \int_U g(s,a)\, \delta_{u_s^n}(da) \underset{n\longrightarrow\infty}{\longrightarrow} \int_0^T \int_U g(s,a)\, q_s(da).$$

Then

$$\int_0^t \int_U f(s,a)\, \delta_{u_s^n}(da) \underset{n\longrightarrow\infty}{\longrightarrow} \int_0^t \int_U f(s,a)\, q_s(da).$$

The set $\{(s,t)\ ;\ 0 \leq s \leq t \leq T\}$ generate $\mathcal{B}_{[0,T]}$. Then $\forall B \in \mathcal{B}_{[0,T]}$, we have

$$\int_B \int_U f(s,a)\, \delta_{u_s^n}(da) \underset{n\longrightarrow\infty}{\longrightarrow} \int_B \int_U f(s,a)\, q_s(da).$$

This implies that

$$\int_U f(s,a)\, \delta_{u_s^n}(da) \underset{n\longrightarrow\infty}{\longrightarrow} \int_U f(s,a)\, q_s(da)\ \ ,\ \ dt - a.e.$$

The lemma is proved. ■

The next lemma gives the stability of the controlled FBSDE with respect to the control variable.

Lemma 14 *Let $q \in \mathcal{R}$ be a relaxed control and (y^q, z^q) the corresponding trajectory. Then there exists a sequence $(u^n)_n \subset \mathcal{U}$ such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left|y_t^n - y_t^q\right|^2\right] = 0, \tag{34}$$

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \left|z_t^n - z_t^q\right|^2 dt = 0, \tag{35}$$

$$\lim_{n\to\infty} J(u^n) = J(q). \tag{36}$$

where (y^n, z^n) denotes the solution of equation (1) associated with u^n.

Proof. i) Proof of (33) and (34).

We have

$$\left\{\begin{array}{ll} d\left(y_t^n - y_t^q\right) = & -\left[b\left(t, y_t^n, z_t^n, u_t^n\right) - b\left(t, y_t^q, z_t^q, u_t^n\right)\right] dt \\ & -\left[b\left(t, y_t^q, z_t^q, u_t^n\right) - b\left(t, y_t^q, z_t^q, u_t^n\right)\right] dt \\ & -\left[b\left(t, y_t^q, z_t^q, u_t^n\right) - \displaystyle\int_U b\left(t, y_t^q, z_t^q, a\right) q_t\left(da\right)\right] dt \\ & +\left(z_t^n - z_t^q\right) dW_t, \\ y_T^n - y_T^q = & 0. \end{array}\right.$$

Put

$$\begin{aligned} Y_t^n &= y_t^n - y_t^q, \\ Z_t^n &= z_t^n - z_t^q, \end{aligned}$$

and

$$\begin{aligned} \varphi^n\left(t, Y_t^n, Z_t^n\right) = b\left(t, y_t^q, z_t^q, u_t^n\right) - \int_U b\left(t, y_t^q, z_t^q, a\right) q_t\left(da\right) \\ + \int_0^1 b_y\left(t, y_t^q + \lambda\left(y_t^n - y_t^q\right), z_t^q + \lambda\left(z_t^n - z_t^q\right), u_t^n\right) Y_t^n d\lambda \\ + \int_0^1 b_z\left(t, y_t^q + \lambda\left(y_t^n - y_t^q\right), z_t^q + \lambda\left(z_t^n - z_t^q\right), u_t^n\right) Z_t^n d\lambda. \end{aligned} \tag{37}$$

Then

$$\left\{\begin{array}{l} dY_t^n = \varphi^n\left(t, Y_t^n, Z_t^n\right) dt + Z_t^n dW_t, \\ Y_T^n = 0. \end{array}\right. \tag{38}$$

The above equation is a linear BSDE with bounded coefficients and with terminal condition $Y_T^n = 0$. Then by applying a priori estimates (see Briand et al [8, Proposition 3.2, Page 7]), we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Y_t^n\right|^2 + \int_0^T \left|Z_t^n\right|^2 dt\right] \leq C\mathbb{E}\left|\int_0^T \left|\varphi^n\left(t, 0, 0\right)\right| dt\right|^2.$$

From (36), we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Y_t^n\right|^2 + \int_0^T \left|Z_t^n\right|^2 dt\right] \leq C\mathbb{E}\left|\int_0^T \left|b\left(t, y_t^q, z_t^q, u_t^n\right) - \int_U b\left(t, y_t^q, z_t^q, a\right) q_t\left(da\right)\right| dt\right|^2.$$

Since b is continuous and bounded, then from (32) and the dominated convergence theorem, the term in the right hand side tends to zero as n tends to infinity. This prove (33) and (34).

ii) Proof of (35).

Since g and h are uniformly Lipshitz with respect to (y,z), then by using the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
&\left|J\left(q^{n}\right)-J\left(q\right)\right| \\
&\leq C\left(\mathbb{E}\left|y_{0}^{n}-y_{0}^{q}\right|^{2}\right)^{1/2}+C\left(\int_{0}^{T}\mathbb{E}\left|y_{t}^{n}-y_{t}^{q}\right|^{2}ds\right)^{1/2}+C\left(\mathbb{E}\int_{0}^{T}\left|z_{t}^{n}-z_{t}^{q}\right|^{2}dt\right)^{1/2} \\
&+C\left(\mathbb{E}\left|\int_{0}^{T}h\left(t,y_{t}^{q},z_{t}^{q},u_{t}^{n}\right)dt-\int_{0}^{T}\int_{U}h\left(t,y_{t}^{q},z_{t}^{q},a\right)q_{t}\left(da\right)dt\right|^{2}\right)^{1/2}.
\end{aligned}
$$

From (33) and (34) the first three terms in the right hand side converge to zero. Furthermore, since h is continuous and bounded, then from (32) and by using the dominated convergence theorem, the fourth term in the right hand side tends to zero. This prove (35). ∎

Lemma 15 *As a consequence of* (35)*, the strict and the relaxed control problems have the same value functions. That is*

$$\inf_{v\in\mathcal{U}}J\left(v\right)=\inf_{q\in\mathcal{R}}J\left(q\right). \tag{39}$$

Proof. Let $u\in\mathcal{U}$ and $\mu\in\mathcal{R}$ be respectively a strict and relaxed controls such that

$$J\left(\mu\right)=\inf_{q\in\mathcal{R}}J\left(q\right). \tag{40}$$

$$J\left(u\right)=\inf_{v\in\mathcal{U}}J\left(v\right), \tag{41}$$

By (39), we have

$$J\left(\mu\right)\leq J\left(q\right),\ \forall q\in\mathcal{R}.$$

Since $\delta\left(\mathcal{U}\right)\subset\mathcal{R}$, then

$$J\left(\mu\right)\leq J\left(q\right),\ \forall q\in\delta\left(\mathcal{U}\right).$$

Since $q\in\delta\left(\mathcal{U}\right)$, then $q=\delta_{v}$, where $v\in\mathcal{U}$. It follows

$$\begin{cases}\left(y^{q},z^{q}\right)=\left(y^{v},z^{v}\right), \\ J\left(q\right)=J\left(v\right).\end{cases}$$

Hence,

$$J\left(\mu\right)\leq J\left(v\right),\ \forall v\in\mathcal{U}.$$

The control u becomes an element of $\mathcal{U}$, then we get

$$J\left(\mu\right)\leq J\left(u\right). \tag{42}$$

On the other hand, by (40) we have

$$J\left(u\right)\leq J\left(v\right),\ \forall v\in\mathcal{U}. \tag{43}$$

The control μ becomes a relaxed control, then by lemma 12, there exists a sequence $(v^n)_n$ of strict controls such that

$$dt\mu_t^n(da) = dt\delta_{v_t^n}(da) \underset{n \longrightarrow \infty}{\longrightarrow} dt\mu_t(da) \text{ stably}, \quad \mathcal{P} - a.s.$$

By (42), we get then

$$J(u) \leq J(v^n), \forall n \in \mathbb{N},$$

By using (35) and letting n go to infinity in the above inequality, we get

$$J(u) \leq J(\mu). \tag{44}$$

Finally, by (41) and (43), we have

$$J(u) = J(\mu).$$

The lemma is proved. ■

To establish necessary optimality conditions for strict controls, we need the following lemma

Lemma 16 *The strict control u minimizes J over $\mathcal{U}$ if and only if the relaxed control $\mu = \delta_u$ minimizes J over $\mathcal{R}$.*

Proof. Suppose that u minimizes the cost J over $\mathcal{U}$, then

$$J(u) = \inf_{v \in \mathcal{U}} J(v).$$

By (38), we get

$$J(u) = \inf_{q \in \mathcal{R}} J(q).$$

Since $\mu = \delta_u$, then

$$\begin{cases} (y^\mu, z^\mu) = (y^u, z^u), \\ J(\mu) = J(u), \end{cases} \tag{45}$$

This implies that

$$J(\mu) = \inf_{q \in \mathcal{R}} J(q).$$

Conversely, if $\mu = \delta_u$ minimize J over $\mathcal{R}$, then

$$J(\mu) = \inf_{q \in \mathcal{R}} J(q).$$

By (38), we get

$$J(\mu) = \inf_{v \in \mathcal{U}} J(v).$$

And By (44), we obtain

$$J(u) = \inf_{v \in \mathcal{U}} J(v).$$

The proof is completed. ■

The following lemma, who will be used to establish sufficient optimality conditions for strict controls, shows that we get the results of the above lemma if we replace $\mathcal{R}$ by $\delta(\mathcal{U})$.

Lemma 17 *The strict control u minimizes J over $\mathcal{U}$ if and only if the relaxed control $\mu = \delta_u$ minimizes J over $\delta(\mathcal{U})$.*

Proof. Let $\mu = \delta_u$ be an optimal relaxed control minimizing the cost J over $\delta(\mathcal{U})$, we have then

$$J(\mu) \leq J(q), \quad \forall q \in \delta(\mathcal{U}).$$

Since $q \in \delta(\mathcal{U})$, then there exists $v \in \mathcal{U}$ such that $q = \delta_v$.
It is easy to see that

$$\begin{cases} (y^\mu, z^\mu) = (y^u, z^u), \\ (y^q, z^q) = (y^v, z^v), \\ J(\mu) = J(u), \\ J(q) = J(v). \end{cases} \tag{46}$$

Then, we get

$$J(u) \leq J(v), \quad \forall v \in \mathcal{U}.$$

Conversely, let u be a strict control minimizing the cost J over $\mathcal{U}$. Then

$$J(u) \leq J(v), \quad \forall v \in \mathcal{U}.$$

Since the controls $u, v \in \mathcal{U}$, then there exist $\mu, q \in \delta(\mathcal{U})$ such that

$$\mu = \delta_u \quad , \quad q = \delta_v.$$

This implies that relations (45) hold. Consequently, we get

$$J(\mu) \leq J(q), \quad \forall q \in \delta(\mathcal{U}).$$

The proof is completed. ■

4.1 Necessary optimality conditions for strict controls

Define the Hamiltonian H in the strict case from $[0,T] \times \mathbb{R}^n \times \mathcal{M}_{n \times d}(\mathbb{R}) \times \mathbb{R}^n \times U$ into $\mathbb{R}$ by

$$H(t, y, z, p, v) = pb(t, y, z, v) - h(t, y, z, v).$$

Theorem 18 *(Necessary optimality conditions for strict controls) Let u be an optimal control minimizing the functional J over $\mathcal{U}$ and (y_t^u, z_t^u) the solution of* (1) *associated with u. Then there an unique adapted process p, solution of*

$$\begin{cases} dp_t^u = H_y\left(t, y_t^u, z_t^u, u_t, p_t^u\right) dt + H_z\left(t, y_t^u, z_t^u, u_t, p_t^u\right) dW_t, \\ p_0^u = h_y(y_0^u). \end{cases} \tag{47}$$

Such that

$$H\left(t, y_t^u, z_t^u, u_t, p_t^u\right) \geq H\left(t, y_t^u, z_t^u, v_t, p_t^u\right); \forall v_t \in U; \ ae\ ,\ as. \tag{48}$$

Proof. Let u be an optimal solution of the strict control problem $\{(1), (2), (3)\}$. Then, there exist $\mu \in \delta(\mathcal{U})$ such that

$$\mu = \delta_u.$$

Since u minimizes the cost J over $\mathcal{U}$, then by lemma 15, μ minimizes J over $\mathcal{R}$. Hence, by the necessary optimality conditions for relaxed controls (Theorem 10), there exist an unique adapted process p^μ, solution of (25), such that

$$\mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) \geq \mathcal{H}\left(t, y_t^\mu, z_t^\mu, q_t, p_t^\mu\right), \ \forall q_t \in \mathbb{P}(U), \ a.e,\ a.s. \tag{49}$$

Since $\delta(U) \subset \mathbb{P}(U)$, then we get

$$\mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) \geq \mathcal{H}\left(t, y_t^\mu, z_t^\mu, q_t, p_t^\mu\right), \ \forall q_t \in \delta(U), \ a.e,\ a.s. \tag{50}$$

Since $q \in \delta(\mathcal{U})$, then there exist $v \in \mathcal{U}$ such that $q = \delta_v$.
We note that v is an arbitrary element of $\mathcal{U}$ since q is arbitrary.
Now, since $\mu = \delta_u$ and $q = \delta_v$, we can easily see that

$$\begin{cases} \left(y^\mu, z^\mu\right) = \left(y^u, z^u\right), \\ \left(y^q, z^q\right) = \left(y^v, z^v\right), \\ p^\mu = p^u, \\ \mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) = H\left(t, y_t^u, z_t^u, u_t, p_t^u\right), \\ \mathcal{H}\left(t, y_t^\mu, z_t^\mu, q_t, p_t^\mu\right) = H\left(t, y_t^u, z_t^u, v_t, p_t^u\right), \end{cases} \tag{51}$$

where p^u is the unique solutions of (46).

By using (49) and (50), we can easy deduce (47). The proof is completed. ■

4.2 Sufficient optimality conditions for strict controls

We recall assumptions (4) an the adjoint equation (46).

Theorem 19 *(Sufficient optimality conditions for strict controls) Assume that g is convex and the function $(y, z) \longmapsto H(t, y, z, q, p)$ is concave. Then, u is an optimal solution of the control problem $\{(1), (2), (3)\}$, if it satisfies* (47) *.*

Proof. Let u be a strict control (candidate to be optimal) such that necessary optimality conditions for strict controls hold. That is

$$H\left(t, y_t^u, z_t^u, u_t, p_t^u\right) \geq H\left(t, y_t^u, z_t^u, v_t, p_t^u\right), \forall v_t \in U, \ a.e, \ a.s. \tag{52}$$

The controls u, v are elements of $\mathcal{U}$, then there exist $\mu, q \in \delta\left(\mathcal{U}\right)$ such that

$$\mu = \delta_u \quad , \quad q = \delta_v.$$

This implies that relations (50) hold, then by (51), we deduce that

$$\mathcal{H}\left(t, y_t^\mu, z_t^\mu, \mu_t, p_t^\mu\right) \geq \mathcal{H}\left(t, y_t^\mu, z_t^\mu, q_t, p_t^\mu\right), \ \forall q_t \in \delta\left(U\right), \ a.e, \ a.s.$$

Since H is concave in (y, z), it is easy to see that $\mathcal{H}$ is concave in (y, z), and since g is convex, then by the same proof that in theorem 11, we show that μ minimizes the cost J over $\delta\left(\mathcal{U}\right)$.

Finally by lemma 16, we deduce that u minimizes the cost J over $\mathcal{U}$. The proof is completed. ■

Remark 20 *The sufficient optimality conditions for strict controls are proved without assuming neither the convexity of U nor that of H in v.*

References

[1] S. Bahlali, B. Mezerdi and B. Djehiche, *Approximation and optimality necessary conditions in relaxed stochastic control problems,* Journal of Applied Mathematics and Stochastic Analysis, Volume 2006, pp 1-23.

[2] S. Bahlali and B. Labed, *Necessary and sufficient conditions of optimality for optimal control problem with initial and terminal costs*, Rand. Operat. and Stoch. Equ, 2006, Vol 14, No3, pp 291-301.

[3] S. Bahlali, B. Djehiche and B. Mezerdi, *The relaxed maximum principle in singular control of diffusions*, SIAM J. Control and Optim, 2007, Vol 46, Issue 2, pp 427-444.

[4] S. Bahlali, *Necessary and sufficient conditions of optimality for relaxed and strict control problems*, SIAM J. Control and Optim, 2008, Vol. 47, No. 4, pp. 2078–2095.

[5] S. Bahlali, *Necessary and sufficient optimality conditions for relaxed and strict control problems of forward-backward systems*, SIAM J. on Applied Mathematics, Submitted.

[6] S. Bahlali, *Necessary and sufficient condition of optimality for optimal control problem of forward and backward systems*, Theory of Probability and Its Applications (TVP), In revision.

[7] A. Bensoussan, *Non linear filtering and stochastic control.* Proc. Cortona 1981, Lect. notes in Math. 1982, 972, Springer Verlag.

[8] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, *L^p Solutions of backward stochastic differential equations*, Sochastic Process and their Applications, No 108, 2003, pp 109-129.

[9] N. Dokuchaev and X. Y. Zhou, *Stochastic controls with terminal contingent conditions,* Journal Of Mathematical Analysis And Applications, 1999, 238, pp 143-165.

[10] N. El Karoui, N. Huu Nguyen and M. Jeanblanc Piqué, *Compactification methods in the control of degenerate diffusions.* Stochastics, Vol. 20, 1987, pp 169-219.

[11] N. El Karoui and L. Mazliak, *Backward stochastic differential equations*, 1997, Addison Wesley, Longman.

[12] N. El-Karoui, S. Peng, and M. C. Quenez, *Backward stochastic differential equations in finance*, 1997, Math. finance 7.

[13] N. El-Karoui, S. Peng and M.C. Quenez, *A dynamic maximum principle for the optimization of recursive utilities under constraints*, Annals of Applied Probability, 11(2001), pp 664-693.

[14] W.H. Fleming, *Generalized solutions in optimal stochastic control*, Differential games and control theory 2, (Kingston conference 1976), Lect. Notes in Pure and Appl. Math.30, 1978.

[15] N.F. Framstad, B. Oksendal and A. Sulem, *A sufficient stochastic maximum principle for optimal control of jump diffusions and applications to finance*, J. Optim. Theory and applications. 121, 2004, pp 77-98.

[16] M. Fuhrman and G. Tessitore, *Existence of optimal stochastic controls and global solutions of forward-backward stochastic differential equations*, SIAM J. Control and Optim, 2004, Vol 43, N° 3, pp 813-830.

[17] U.G. Haussmann, *General necessary conditions for optimal control of stochastic systems*, Math. Programming Studies 6, 1976, pp 30-48.

[18] U.G. Haussmann, *A Stochastic maximum principle for optimal control of diffusions*, Pitman Research Notes in Math, 1986, Series 151.

[19] S.Ji and X. Y. Zhou, *A maximum principle for stochastic optimal control with terminal state constraints, and its applications.* Commun. Inf. Syst, 2006, 6(4), pp 321-338.

[20] H.J. Kushner, *Necessary conditions for continuous parameter stochastic optimization problems*, SIAM J. Control Optim, Vol. 10, 1973, pp 550-565.

[21] J. Ma and J. Yong, *Solvability of forward-backward SDEs and the nodal set of Hamilton-Jacobi-Bellman equations*. A Chinese summary appears in Chinese Ann. Math. Ser. A 16 (1995), no. 4, 532. Chinese Ann. Math. Ser. B 16, 1995, no. 3, pp 279–298.

[22] J. Ma and J. Zhang, *Representation theorems for backward stochastic differential equations*, Ann. Appl. Probab., 2002, 12(4), pp 1390-1418, 2002.

[23] B. Mezerdi and S. Bahlali, *Approximation in optimal control of diffusion processes*, Rand. Operat. and Stoch. Equ, 2000, Vol.8, No 4, pp 365-372.

[24] B. Mezerdi and S. Bahlali, *Necessary conditions for optimality in relaxed stochastic control problems*, Stochastics And Stoch. Reports, 2002, Vol 73 (3-4), pp 201-218.

[25] E. Pardoux and S. Peng, *Adapted solutions of backward stochastic differential equations*, Sys. Control Letters, 1990, Vol. 14, pp 55-61.

[26] S. Peng, *A general stochastic maximum principle for optimal control problems.* SIAM J. Control and Optim. 1990, 28, N° 4, pp 966-979.

[27] S. Peng, *Backward stochastic differential equations and application to optimal control*, Appl. Math. Optim. 1993, 27, pp 125-144.

[28] S. Peng and Z. Wu, *Fully coupled forward-backward stochastic differential equations and applications to optimal control*. SIAM J. Control Optim., 1999, 37, no. 3, pp. 825–843.

[29] J.T. Shi and Z. Wu, *The maximum principle for fully coupled forward-backward stochastic control system*, Acta Automatica Sinica, Vol 32, No 2, 2006, pp 161-169.

[30] Z. Wu, *Maximum Principle for Optimal Control Problem of Fully Coupled Forward-Backward Stochastic Systems*, Systems Sci. Math. Sci, 1998, 11, No.3, pp 249-259.

[31] W. Xu, *Stochastic maximum principle for optimal control problem of forward and backward system,* J. Austral. Math. Soc. Ser. B 37, 1995, pp 172-185.

[32] J. Yong and X.Y. Zhou, *Stochastic controls : Hamilton systems and HJB equations*, vol 43, Springer, New York, 1999.

[33] X.Y. Zhou, *Sufficient conditions of optimality for stochastic systems with controllable diffusions*. IEEE Trans. on Automatic Control, 1996, 41, pp 1176-1179.